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07 December 2012

### Version of attached file:

Published Version

### Peer-review status of attached file:

Not peer-reviewed

### Citation for published item:

Marsiliani, Laura and Renstroem, Thomas (2009) 'Integrated aggregation in dynamic economies.', Working Paper. Durham University, Durham.

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# AGGREGATION IN DYNAMIC ECONOMIES

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The paper provides necessary and sufficient conditions for aggregation of heterogeneous individuals in dynamic economies, when individuals differ in abilities as well as in capital endowments, and when there are distortionary taxes. The aggregation theorems imply that the competitive equilibrium can be represented as if there was only one individual in the economy. This considerably facilitates analysis of the aggregate economy, such as stability analysis, as well as of the distribution of wealth. Furthermore, the paper provides conditions under which a representative individual coincides with one of the individuals in the economy.

*Keywords:* Aggregation, economic dynamics, heterogeneity.

*JEL classification:* D30, D91, E21.

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# 1 INTRODUCTION

Analysis of heterogeneous individuals (e.g. analysis of distribution, welfare, fiscal policy) is considerably facilitated if the prices in the competitive equilibrium are independent of the distributional characteristics on the economy. The question is under what circumstances is this true? This paper provides necessary and sufficient conditions on the individuals' utility functions for aggregation to occur. We analyse an economy which is more complex than previous studies ([Pollack (1971)]). We allow individuals to differ in abilities as well as in capital endowments. Pollack (1971) proved necessary and sufficient conditions when individuals differ in endowments and face the same prices and without a government.<sup>1</sup> Furthermore our proofs of necessity and sufficiency are more direct and transparent than those provided by Pollack, and draws on the explicit dynamic structure of the economy.

The paper is structured as follows. Section 2 states the underlying assumptions, section 3 solves for the economic equilibrium and section 4 provides the aggregation theorems. Section 5 concludes.

## 2 THE ECONOMY

### 2.1 ASSUMPTIONS

#### A1 Individual Preferences

An individual of type  $i$  chooses consumption and labour supply paths,  $c^i(t)$  and  $l^i(t)$  for  $t \in [0, \infty)$  so as to maximise her life-time utility.

$$V(a_0^i) = \max_{c^i, l^i} \int_0^{\infty} e^{-\theta t} u(c^i(t), l^i(t)) dt \quad (1)$$

#### A2 Individuals' Constraints

Individual  $i$  owns (net) assets  $a^i(t)$  and earns interest at a rate  $r(t)$ . For each efficient unit of supplied labour she earns the wage rate  $w(t)$ . The taxes on capital income and labour income are denoted  $\tau^k(t)$  and  $\tau^l(t)$  respectively (in a flat income-tax system they are equal). Define the *after-tax returns*  $\rho(t) \equiv [1 - \tau^k(t)]r(t)$  and  $\omega(t) \equiv [1 - \tau^l(t)]w(t)$ . Denote the productivity of individual  $i$  as  $\gamma^i$ , which we normalise so that the population average is one, i.e.  $\int \gamma^i dF(j) = 1$ . The

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<sup>1</sup> Pollack's economy was static but could easily be reinterpreted as a dynamic one.

individual receives a lump-sum transfer  $b(t)$  from the government, and her budget constraint is therefore

$$\begin{aligned}\dot{a}^i(t) &= \rho(t)a^i(t) + \omega(t)\gamma^i l^i(t) - c^i(t) + b(t) \\ a^i(0) &= a_0^i\end{aligned}\tag{2}$$

### A3 Production

There is a large number of competitive firms in the economy, each of whom operating under *constant-returns-to-scale* technology. Therefore aggregate production,  $y(t)$ , can be calculated as if there was a representative firm employing the aggregate quantities of capital and labour, respectively defined as  $\bar{k}(t) \equiv \int k^i(t) dF(i)$  and  $\bar{l}(t) \equiv \int l^i(t) dF(i)$ . Thus

$$f(\bar{k}, \bar{l}) = f_k(\bar{k}, \bar{l})\bar{k} + f_l(\bar{k}, \bar{l})\bar{l}\tag{3}$$

Let  $\gamma^i$ , the productivity parameter, be normalised such that

$$\int \gamma^i dF(i) = 1\tag{4}$$

where  $F(i)$  is the distribution of individuals in the economy. Also the population is normalised to unity so that the population averages coincide with the population aggregates.

### A4 Public sector

Real public expenditure takes the form of sequences of public goods  $g(t)$  and lump-sum rebates  $b(t)$ ,  $t \in [0, \infty)$ . Denote average quantities with overbar, then the government's constraint is<sup>2</sup>

$$0 = -\tau^k(t)r(t)a(t) - \tau^l(t)w(t)l(t) + g(t) + b(t)\tag{5}$$

The evolution of the capital stock is therefore

$$\begin{aligned}\dot{k}(t) &= f(k(t), l(t)) - c(t) - g(t) \\ k(0) &= k_0\end{aligned}\tag{6}$$

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<sup>2</sup> Because of the lump-sum rebate this economy is Ricardian (public debt neutral), and therefore we simply (without any loss of generality) could have ignored public debt and have imposed budget balance in each period.

### 3 ECONOMIC EQUILIBRIUM

#### 3.1 INDIVIDUAL ECONOMIC BEHAVIOUR

The current-value Hamiltonian for individual  $i$  is

$$H^i = u(c^i(t), l^i(t)) + q^i(t) \{ \rho(t) a^i(t) + \omega(t) \gamma^i l^i(t) - c^i(t) + b(t) \} \quad (7)$$

The first-order conditions

$$u_c(c^i(t), l^i(t)) - q^i(t) = 0 \quad (8)$$

$$u_l(c^i(t), l^i(t)) + q^i(t) \omega(t) \gamma^i - \chi^i(t) = 0 \quad (9)$$

$$\dot{q}^i(t) = [\theta - \rho(t)] q^i(t) \quad (10)$$

describe the individual's choice of  $c^i$  and  $l^i$  as functions of the co-state  $q^i$ , up to the initial value of  $q^i$ , i.e.  $q^i(0)$ . If  $0 \leq l^i(t)$  is binding  $\chi^i(t) < 0$  and (9) becomes

$$u_l(c^i(t), 0) + \gamma^i \omega(t) q^i(t) - \chi^i(t) = 0. \quad (9')$$

If  $l^i(t) \leq L$  is binding  $\chi^i(t) > 0$  and (9) becomes

$$u_l(c^i(t), L) + \gamma^i \omega(t) q^i(t) - \chi^i(t) = 0. \quad (9'')$$

When there is an interior solution for labour supply we have  $\chi^i(t) = 0$ .

The initial value of the co-state (i.e. marginal utility of the state [individual's assets]) is chosen to its lowest possible value subject to the intertemporal budget constraint

$$0 = a^i(0) + \int_0^\infty e^{-\int_0^t \rho(s) ds} [\omega(t) \gamma^i l^i(t) - c^i(t) + b(t)] dt \quad (11)$$

Therefore,  $q^i(0)$  depends on *all* future tax rates and lump-sum rebates.

We should also notice that condition (10) implies that  $q^i(t)/q^j(t)$  is constant  $\forall i, j$ . So the ratios of marginal indirect utilities are constant over time.

Equations (8) and (9) form a system such that  $c^i$  and  $l^i$  may be solved for as functions of  $q^i$  and  $\omega$ . Their partial derivatives are obtained by differentiating through (8) and (9). Assuming an interior solution for  $l^i$  we have

$$\frac{\partial c^i(t)}{\partial \tau^c(t)} = \frac{u_c^i u_{ll}^i}{D^i [1 + \tau^c(t)]} \quad (12)$$

$$\frac{\partial l^i(t)}{\partial \tau^c(t)} = \frac{-u_c^i u_{lc}^i}{D^i [1 + \tau^c(t)]} \quad (13)$$

$$\frac{\partial c^i(t)}{\partial \omega(t)} = -\frac{u_l^i u_{cl}^i}{D^i \omega(t)} \quad (14)$$

$$\frac{\partial l^i(t)}{\partial \omega(t)} = \frac{u_l^i u_{cc}^i}{D^i \omega(t)} \quad (15)$$

$$\frac{\partial c^i(t)}{\partial q^i(t)} = \frac{u_c^i u_{ll}^i - u_l^i u_{cl}^i}{D^i q^i(t)} \quad (16)$$

$$\frac{\partial l^i(t)}{\partial q^i(t)} = \frac{u_l^i u_{cc}^i - u_c^i u_{lc}^i}{D^i q^i(t)} \quad (17)$$

where  $D^i \equiv u_{cc}^i u_{ll}^i - u_{cl}^i u_{lc}^i > 0$ , and  $u_{ll}^i$  is shorthand for  $u_{ll}(c^i(t), l^i(t))$  etc. If there is a corner solution for  $l^i(t)$  (i.e.  $l^i(t) \neq 0$ ) equation (13) and (15) are zero.

The equations (12)-(17) are *compensated changes* in the individual demand functions. For example  $\partial c^i(t)/\partial \tau^c(t)$  is the change in individual consumption when the tax on consumption changes and the individual is compensated with initial capital so as to keep the marginal utility of capital at date  $t$  constant (i.e. keeping  $q^i(t)$  constant). We see that compensated consumption is decreasing in the consumption tax and compensated labour is increasing in the after-tax wage rate. The compensated cross-price effects depend on the sign of  $u_{lc}^i$ , i.e. whether marginal utility of consumption is increasing or decreasing with the amount of labour supplied. If utility is additively separable this term is zero (and the cross-price effects are zero). We see that (16) is negative if consumption is a normal good, and (17) is positive if leisure is a normal good.<sup>3</sup>

### 3.2 AGGREGATE ECONOMIC BEHAVIOUR

The partial derivatives (12)-(17) are all individual specific since the utility function is evaluated at individual specific quantities. The aggregate (or equivalently the mean<sup>4</sup>) behaviour of the economy is given by summing the derivatives over all individuals, or as in this case with a continuum of individuals: integrating the derivatives over all individuals

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<sup>3</sup> It turns out that sufficient for local stability of the economy is that both consumption and leisure are normal goods.

<sup>4</sup> Recall that we have normalised the population to unity, so that the aggregates coincide with the means.

$$\frac{\partial \bar{c}(t)}{\partial \xi(t)} = \int \frac{\partial c^i(t)}{\partial \xi(t)} dF(i), \quad \frac{\partial \bar{l}(t)}{\partial \xi(t)} = \int \gamma^i \frac{\partial l^i(t)}{\partial \xi(t)} dF(i), \quad (18)$$

$$\xi(t) = \{ \omega(t), \tau^c(t), q(t) \}.$$

These aggregate derivatives could be non-linear functions of the individual variables,  $c^i(t)$  and  $l^i(t)$ . Notice that we premultiply the individuals' labour supplies with their respective productivity parameter when we aggregate. This is because we want the aggregate labour supply in efficiency units (not in hours worked). It is only efficient labour supply that generates income which can be redistributed, and only efficient labour supply matters for production.

It is very difficult to keep track of the distribution for the stability analysis (in fact for an  $n$ -consumer economy, we would have a  $2n \times 2n$  dimensional Jacobian matrix with  $2n$  eigen values). Because of this we need to restrict to a special class of preferences that permits *aggregation*, i.e. that permits us to express aggregate quantities as functions of aggregate quantities only. We shall derive necessary and sufficient conditions for aggregation to hold, concentrating on sub-classes of preferences: additively separable and multiplicatively separable. The aggregation theorems that follow are new to the literature. Previously, necessary and sufficient conditions have been proven for additively separable utility when individuals face the same prices (i.e. when they do not differ in productivity). This was done by Pollack (1971). Furthermore, Pollack's way of proving his theorem is much longer and less direct than ours. We shall also prove necessary and sufficient conditions for aggregation when the utility function is multiplicatively separable.

## 4 AGGREGATION

### 4.1 PRELIMINARIES

It will turn out to be the case that the necessary and sufficient conditions for aggregation relate to the family of preferences labelled *Hyperbolic Absolute Risk Aversion* (HARA). We begin by defining the HARA-family, and we use Ingersoll's (1987, p.39) definition:<sup>5</sup>

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<sup>5</sup> See also Merton (1990), p.137.

**Definition** (HARA-Preferences; Ingersoll (1987, p.39)) A utility function  $u: \mathbb{R}_+ \rightarrow \mathbb{R}$  is said to belong to the family Hyperbolic Absolute Risk Aversion (HARA), if it is any positive linear transformation of

$$\tilde{u}(x) = \frac{1}{\delta - 1} \left( a + \delta x \right)^{\frac{\delta - 1}{\delta}} \quad (19)$$

where  $a$  and  $\delta$  are constants, and  $a + \delta x \geq 0$ .

Notice that the HARA function is a function in one argument only. The HARA family is a broad class of preferences that incorporates (as special cases) a number of the most commonly used utility functions. By taking limits of the parameters  $a$  and  $\delta$  and adding constants we can generate these functions from (21). They can be found in Ingersoll (1990, p.39) but for completeness we shall list them here. The special cases listed by Ingersoll (1990) are:

Iso-elastic utility, obtained when  $a=0$  and  $\delta > 0$ .

Logarithmic utility, obtained when  $a=0$  and  $\lim_{\delta \rightarrow 1} \tilde{u} - 1/(\delta - 1)$ .

Negative exponential, obtained when  $a > 0$  and  $\lim_{\delta \rightarrow 0} \tilde{u} - a/\delta$ .

Quadratic, obtained when  $a > x^{\max}$  and  $\delta = -1$ .

However, we see that there is another common function that can be generated as well: Stone-Geary.

Stone-Geary is obtained when  $a = -\delta x^0$  and  $\delta > 0$  [ $x^0$  being interpreted as the subsistence level of consumption].

An equivalent formulation of HARA-preferences is *Linear Risk Tolerance* (LRT):<sup>6,7</sup>

$$-\frac{\tilde{u}'(x)}{\tilde{u}''(x)} = a + \delta x > 0 \quad (20)$$

where  $a$  and  $\delta$  are constants.

The left-hand side of (20) is the inverse of the Arrow-Pratt measure Absolute Risk

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<sup>6</sup> There is no risk here, so we should think of the risk-tolerance measure as a measure of intertemporal substitution, i.e. how willing an individual is to defer consumption to the future.

<sup>7</sup> See Ingersoll (1987), p. 39, and Merton (1990), p.137.



Aversion, and is generally in the finance literature referred to as Risk Tolerance. Equation (20) states that Risk Tolerance is linear in its argument, i.e. *Linear Risk Tolerance* (LRT). Equation (20) can be integrated twice to get the parameterised HARA-utility function (19). Therefore equation (20) is equivalent to any positive linear transformation of (19).

We shall derive the necessary and sufficient conditions for utility functions to have the property that the aggregate derivatives (18) are functions of only aggregate variables. This will be the case if the individual derivatives (12)-(17) are linear functions of the individual  $c^i(t)$  and  $l^i(t)$ . All utility functions are a too broad class to work with. Therefore we will concentrate on *separable preferences*: either additively separable or multiplicatively separable. So we shall find necessary and sufficient conditions among separable preferences that permit aggregation. From these conditions we can find the functional forms. These functional forms we shall define as a class of preferences, labelled  $\bar{U}$ , which is necessary and sufficient for aggregation (among separable preferences).

## 4.2 AGGREGATION THEOREMS

As mentioned previously, the only necessary and sufficient conditions for aggregation, when individuals face the same prices (i.e. do not differ in productivity) and when they have additively separable utility was obtained by Pollack (1971). His proof was much longer and less direct than the one we are going to provide.<sup>8</sup> In the theorems we provide we have found both necessity and sufficiency and allow for differences in productivity. Furthermore the class we analyse is also broader than Pollack, since he allowed only for additively separable preferences.

**Theorem 1** *Assume A1, A2 and A3, and any price paths  $\rho(t)$ ,  $\omega(t)$ ,  $\tau^c(t)$ ,  $t \in [0, \infty)$ , such that an equilibrium exists where  $0 < l^i(t) < L$ ,  $\forall i$ . Then if  $u$  is additively separable in  $c$  and  $l$ , i.e.*

$$u = \Phi(c) + \Psi(L-l),$$

*necessary and sufficient for aggregate consumption, aggregate labour supply, and aggregate capital stock being functions of only aggregate variables is that both  $\Phi$  and  $\Psi$  belong to the*

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<sup>8</sup> He proves sufficiency and necessity for three types of utility functions: logarithmic, negative exponential, and a third function which actually is HARA (but he never refer to it as HARA, though the HARA-family was known at that time). Instead he could have proved it for his third function and derived the other two (logarithmic and negative exponential) as special cases.

HARA-family.

**Theorem 2** Given A1, A2 and A3, and any price paths  $\rho(t)$ ,  $\omega(t)$ ,  $\tau^c(t)$ ,  $t \in [0, \infty)$ , such that an equilibrium exists where  $0 < l^i(t) < L$ ,  $\forall i$ . Then if  $u$  is multiplicatively separable in  $c$  and  $l$ , i.e.

$$u = \Phi(c)\Psi(L-l),$$

necessary and sufficient for aggregate consumption, aggregate labour supply, and aggregate capital stock being functions of only aggregate variables is that the functions  $\Phi$  and  $\Psi$  take the form

$$\Phi(c) = \frac{1}{\delta-1} \left( a + \delta c \right)^{\frac{\delta-1}{\delta}}, \quad \Psi(L-l) = \frac{1}{\eta-1} \left( b + \eta(L-l) \right)^{\frac{\eta-1}{\eta}} \quad (21)$$

where  $(\delta-1)(\eta-1) < 1$ . That is  $\Phi$  and  $\Psi$  are restricted versions of the HARA-family.

Proof of Theorem 1 and 2: We need to show that the partial derivatives of the aggregate consumption and aggregate labour supply functions are functions only of aggregate variables. The derivative of the aggregate is the aggregate of the individual derivatives (equation (18)). We shall prove Theorem 1 first. When utility is additively separable, as in Theorem 1, the cross derivative  $u_{cl}^i$  is zero and therefore  $D^i = u_{cc}^i u_{ll}^i$  and we have  $\partial \bar{l}(t) / \partial \tau^c(t) = \partial \bar{c}(t) / \partial \omega(t) = 0$  from (13) and (14) respectively. The derivatives (12), (15)-(17) become proportional either to  $u_c^i / u_{cc}^i$  or to  $u_l^i / u_{ll}^i$ . For the integrals of these ratios to be functions of aggregate consumption and labour supply only, we need these ratios to be linear in individual consumption and labour supply, i.e. we need

$$-u_c^i / u_{cc}^i = a + \delta c^i(t) \text{ and } u_l^i / u_{ll}^i = b + \eta(L-l^i)$$

for some constants  $a$ ,  $b$ ,  $\delta$ ,  $\eta$ . But, this is exactly the definition of Linear Risk Tolerance, which integrated twice gives the HARA-family. This proves Theorem 1.

To prove Theorem 2 we proceed in the same way. To prove necessity we shall concentrate on  $\partial \bar{c}(t) / \partial \tau^c(t)$  and  $\partial \bar{c}(t) / \partial \omega(t)$ . By (12) and (13) we see that it is necessary that  $u_c^i u_{ll}^i / D^i$  and  $u_l^i u_{cl}^i / D^i$  are linear in  $c^i$  and  $l^i$ . Since  $u^i = \Phi(c^i)\Psi(L-l^i)$  we have  $D^i = \Phi\Psi\Phi''\Psi'' - [\Phi'\Psi']^2$ . Then we have

$$\frac{u_c^i u_{ll}^i}{D^i} = \frac{\frac{\Phi'}{\Phi''}}{\frac{\Psi}{\Psi'} \frac{\Psi''}{\Psi'} - \frac{\Phi'}{\Phi} \frac{\Phi'}{\Phi''}} \frac{\Psi}{\Psi'} \frac{\Psi''}{\Psi'} \quad (22)$$

$$\frac{u_l^i u_{cl}^i}{D^i} = \frac{\frac{\Phi'}{\Phi''}}{\frac{\Psi}{\Psi'} \frac{\Psi''}{\Psi'} - \frac{\Phi'}{\Phi} \frac{\Phi'}{\Phi''}} \quad (23)$$

Equation (23) must be linear in  $c^i$  and  $l^i$ , then for (22) also being linear the term  $\Psi\Psi''/[\Psi']^2$  has to be constant, equal to say  $m$ . That is  $\Psi''/\Psi' = m\Psi'/\Psi$ , or by integrating once  $\Psi' = h\Psi$ ,  $h > 0$ , which has the solution

$$\Psi(L-l^i) = [g + h(1-m)(L-l^i)]^{1/(1-m)}$$

This gives  $\Psi$  in (21) (up to a multiplicative constant) if we set  $m = 1/(1-\eta)$  and  $g = hb/(\eta-1)$ . Next, since  $\Psi\Psi''/[\Psi']^2$  is a constant, (23) requires  $\Phi\Phi''/[\Phi']^2$  to be constant as well. This gives  $\Phi$  in (21). Thus necessity is proven. Sufficiency follows since the denominator in (22) and (23) is constant and the numerators are linear in their respective arguments by Linear Risk Tolerance. Finally, given the forms (21) we need to show that (14)-(17) are linear in their arguments. (14) follows immediately since (14) is the sum of (22) and (23). The rest of the derivatives [(15)-(17)] follow by symmetry (changing  $c$  for  $l$  and vice versa).  $\square$

An alternative proof, and almost identical, may be constructed as follows. Differentiating (8)-(9) with respect to time and substituting for (10) gives Euler equations for  $c^i(t)$  and  $l^i(t)$ .

$$\dot{c}^i(t) = \frac{u_c^i u_{ll}^i}{D^i} \left( \theta - \rho(t) + \frac{\dot{\tau}^c(t)}{1 + \tau^c(t)} \right) - \frac{u_l^i u_{cl}^i}{D^i} \left( \theta - \rho(t) + \frac{\dot{\omega}(t)}{\omega(t)} \right) \quad (24)$$

$$\dot{l}^i(t) = -\frac{u_c^i u_{cl}^i}{D^i} \left( \theta - \rho(t) + \frac{\dot{\tau}^c(t)}{1 + \tau^c(t)} \right) + \frac{u_l^i u_{cc}^i}{D^i} \left( \theta - \rho(t) + \frac{\dot{\omega}(t)}{\omega(t)} \right) \quad (25)$$

The derivatives (12)-(17) appear as coefficients in front of the prices. Necessary and sufficient for aggregation of these Euler equations is therefore that the necessary and sufficient conditions for aggregation of these coefficients (the derivatives (12)-(17)). Then the proof of

Theorem 1 and 2 apply directly.

The intuition for the aggregation results can be drawn from HARA-preferences. HARA preferences have linear income expansion paths (linear Engle curves). Any individual would retain the same proportions of the consumption goods at different income levels. Thereby the relative prices in such an economy would be the same independent of the distribution of income.

We shall define a class of preferences,  $\bar{U}$ , that incorporates the utility functions that permit aggregation.

**Definition 2** A utility function  $u: \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is said to belong to  $\bar{U}$  if  $u$  satisfies the conditions in either Theorem 1 or in Theorem 2.

The aggregation theorems imply that we can describe the aggregate quantities as functions of aggregate quantities only. That is

**Corollary 1** Assume A1, A2 and A3, and any price paths  $p(t), \omega(t), \tau^c(t), t \in [0, \infty)$ , such that an equilibrium exists where  $0 < l^i(t) < L, \forall i$ . Then if  $u \in \bar{U}$  there exists a variable  $q(t)$  that follows

$$\dot{q}(t) = (\theta - \rho(t))q(t) \quad (26)$$

such that aggregate consumption, aggregate labour supply, and aggregate capital stock can be represented as

$$\bar{c}(t) = \bar{c}(\omega(t), \tau^c(t), q(t)) \quad (27)$$

$$\bar{l}(t) = \bar{l}(\omega(t), \tau^c(t), q(t)) \quad (28)$$

$$\dot{\bar{k}}(t) = f(\bar{k}(t), \bar{l}(\omega(t), \tau^c(t), q(t))) - \bar{c}(\omega(t), \tau^c(t), q(t)) \quad (29)$$

Proof: Follows from Theorem 1 and 2 and Definition 2.

QED

The average capital accumulation equation (29), and the transition equation (26) for the average co-state variable  $q_t$ , constitute the *aggregate dynamic behaviour* of the economy.

Theorem 1 and 2 are aggregation theorems. They state that the economy behaves as if there was a representative individual. It does not say that there is a particular person in the heterogeneous population that behaves as the representative individual. This, however, can be established in the next theorem, by addition of two extra assumptions: purely redistributive taxation (A4) and equal capital endowments (A5).

#### **A4 Redistributive Taxation**

The tax receipts at time  $t$  are fully redistributed lump sum,  $b(t)$ , to the individuals.<sup>9</sup>

#### **A5 Equal Initial Capital**

Each individual  $i$  is endowed with the same amount of initial capital,  $k_0^i = \bar{k}_0$ .

**Theorem 3** *Given A1, A2, A3, A4 and A5, and any price paths  $p(t)$ ,  $\omega(t)$ ,  $\tau^c(t)$ ,  $t \in [0, \infty)$ , such that an equilibrium exists where  $0 < l^i(t) < L$ ,  $\forall i$ . Then if  $u \in \bar{U}$ , an individual's optimal choice of consumption and labour supply will coincide with the average consumption and average labour supply if and only if he is endowed with the mean skills:  $\gamma = 1$ .*

Proof: Evaluating (11) when initial endowments are equal ( $\Delta^i(0)=0$ ) we have

$$0 = \int_0^\infty e^{-\int_0^t \rho(s) ds} [\omega(t)(\gamma^i l^i(t) - \bar{l}(t)) - [1 + \tau^c](c^i(t) - \bar{c}(t))] dt \quad (30)$$

for all  $i$ . For any individual who chooses the aggregate quantities throughout the entire period this condition becomes

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<sup>9</sup> Without loss of generality we impose period-by-period budget balance, because this economy is *Public-Debt Neutral*. Public-Debt Neutrality (or Ricardian Equivalence [Barro (1974)]) holds because of the lump-sum transfer. It does not matter for the individuals whether the lump sum transfer is redistributed immediately or at some date later on, since individuals can borrow and lend freely. Only the levels of the tax rates matter.

$$0 = \int_0^{\infty} e^{-\int_0^t \rho(s) ds} \omega(t) (\gamma^i - 1) dt \quad (31)$$

and can only hold for  $\gamma^i = 1$ . This proves necessity, i.e. if an individual is observed to have chosen the average quantities, then he must be endowed with the average skills.

To prove sufficiency, denote (30) as an implicit function  $G(\gamma^i, q^i(0)) = 0$ . Then  $\partial q^i(0)/\partial \gamma^i = -G_{\gamma^i}/G_{q^i}$ . First, noticing that  $\partial q^i(t)/\partial q^i(0) = q^i(t)/q^i(0)$ , we have

$$G_{q^i} = \int_0^{\infty} e^{-\int_0^t \rho(s) ds} \left[ \omega(t) \gamma^i \frac{\partial l^i(t)}{\partial q^i(t)} - [1 + \tau^c] \frac{\partial c^i(t)}{\partial q^i(t)} \right] \frac{q^i(t)}{q^i(0)} dt \quad (32)$$

Second,

$$G_{\gamma^i} = \int_0^{\infty} e^{-\int_0^t \rho(s) ds} \left[ \omega(t) \left( l^i(t) + \gamma^i \frac{\partial l^i(t)}{\partial \gamma^i} \right) - [1 + \tau^c] \frac{\partial c^i(t)}{\partial \gamma^i} \right] dt \quad (33)$$

Next, differentiating through (8)-(9) with respect to  $\gamma^i$  (keeping  $q^i(t)$  constant), we obtain

$$\frac{\partial c^i(t)}{\partial \gamma^i} = - \frac{u_l^i u_{cl}^i}{D^i \gamma^i} \quad (34)$$

$$\frac{\partial l^i(t)}{\partial \gamma^i} = \frac{u_l^i u_{cc}^i}{D^i \gamma^i} \quad (35)$$

where  $D^i \equiv u_{cc}^i u_{ll}^i - u_{cl}^i u_{lc}^i > 0$ .

Then  $\omega[l^i + \gamma^i \partial l^i / \partial \gamma^i] - (1 + \tau^c) \partial c^i / \partial \gamma^i = \omega[l^i + u_{cc}^i u_{ll}^i (D^i)^{-1}] + (1 + \tau^c) u_{cl}^i u_{lc}^i (\gamma^i D^i)^{-1} = \omega[l^i + (u_{cc}^i u_{ll}^i - u_{cl}^i u_{lc}^i) (D^i)^{-1}]$ , where the last line follows from (8). Using this in (33)

$$G_{\gamma^i} = \int_0^{\infty} e^{-\int_0^t \rho(s) ds} \omega(t) \left[ l^i(t) + [u_{cc}^i(t) u_{ll}^i(t) - u_{cl}^i(t) u_{lc}^i(t)] [D^i(t)]^{-1} \right] dt \quad (33')$$

Sufficient for this integral to be positive is that leisure in each period is a normal good, i.e.  $u_{cc}^i u_{ll}^i - u_{cl}^i u_{lc}^i \geq 0$ . For the integral in (32) to be positive it is sufficient that both leisure and consumption are normal goods in each period. Then (32) and (33') give  $\partial q^i(0)/\partial \gamma^i < 0$ , so that there is a one-to-one (negative) relation between the individual's skill and her marginal utility of initial capital. By Corollary 1 there exist an initial  $q(0)$  [and a path following (26)] such that the demand functions, (8)-(9), of this  $q$  coincide with the aggregate consumption and

labour supply, (27)-(28). Since there is a one-to-one relation between  $\gamma^i$  and  $q^i(0)$ , the  $q(0)$  that gives the aggregate behaviour must be generated by  $\gamma = 1$  (by (31)).

## 4 CONCLUSIONS

We have found necessary and sufficient conditions for aggregation in a dynastic economy when individuals differ in abilities, for two sub-classes of preferences: additively separable and multiplicatively separable. We derived the closed-form solutions for these two classes, and showed that the closed-form solutions are versions of the HARA-family. The aggregation theorems (Theorem 1 and 2) ensured that the aggregate economy behaves as if there was only one individual in the whole economy, but did not say anything about who this individual is. However, if individuals have the same initial capital endowments and taxes are purely redistributive, then the representative consumer can be identified as the individual with mean skill (Theorem 3).

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